

## Coherence and anticoherence resonance tuned by noise

A. M. Lacasta,<sup>1</sup> F. Sagués,<sup>2</sup> and J. M. Sancho<sup>3</sup>

<sup>1</sup>*Departament de Física Aplicada, Universitat Politècnica de Catalunya, Avenue Gregorio Marañón 44, E-08034 Barcelona, Spain*

<sup>2</sup>*Departament de Química Física, Universitat de Barcelona, Diagonal 647, E-08028 Barcelona, Spain*

<sup>3</sup>*Departament d'Estructura i Constituents de la Matèria, Facultat de Física, Universitat de Barcelona, Diagonal 647, E-08028 Barcelona, Spain*

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We present numerical evidence and a theoretical analysis of the appearance of anticoherence resonance induced by noise, not predicted in former analysis of coherence resonance. We have found that this phenomenon occurs for very small values of the intensity of the noise acting on an excitable system, and we claim that this is a universal signature of a nonmonotonous relaxational behavior near its oscillatory regime. Moreover, we demonstrate that this new phenomenon is totally compatible with the standard situation of coherence resonance appearing at intermediate values of noise intensity.

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Subtle signatures of the ordering role of noise in complex systems constitutes nowadays a new and celebrated paradigm in nonlinear science. Very well documented scenarios are those of stochastic resonance [1] and, closer to our interest here, coherence resonance either in single units [2] or arrays [3]. In this later situation, features of coherent noise-induced emitted pulses can be recognized in autonomous excitable systems that do not exhibit self-sustained oscillations. The whole phenomenon rests on the fact that the time interval between pulses decomposes into an activation and an excursion time which depend differently on the noise amplitude. Restricting to small noise intensities  $D$ , i.e., far indeed from the optimal conditions of stochastic resonance, the first contribution largely exceeds the second one, and thus constitutes the emitting rate controlling process mediated by a Kramers-like mechanism [2]. This in turn leads to a nearly Poisson-like distribution of interspike intervals  $t_p$ . Consequently, taking its normalized variance defined by

$$R = \frac{\sqrt{\langle (\Delta t_p)^2 \rangle}}{\langle t_p \rangle}, \quad (1)$$

as an indicator of a noncoherence emission, we have  $R < 1$ , with the limiting Poissonian value  $R = 1$  approached from below as  $D \rightarrow 0$ .

Apparently thus, there should be nothing specially relevant in this limit for an excitable system subjected to noise. However, and quite surprisingly we will show in this paper that this is not always true. In a very abstract way, the key point to realize is that by referring to a conveniently modified Poisson distribution, one might have a situation opposed to that of coherence resonance, and so termed *anticoherence resonance*, with  $R > 1$  and indeed approaching unity from above. This phenomenon is related with the appearance of another temporal scale which enchains pulses into time intervals much shorter than the mean excitation time. In this way the statistics of pulse appearance would be modified, decreasing both the variance and, hopefully more markedly, the averaged value of the interspike intervals distribution. The existence of situations with  $R > 1$  was conjectured and studied within the context of neuronal dynamics [4,5], but

our emphasis here will be on the fact that the whole phenomenon of coherence and anticoherence is simply controlled by the noise intensity.

We propose that the physical mechanism underlying this phenomenon is that the excitability threshold changes in time during the nonmonotonous but rather oscillatory relaxation to the rest state, as we will see. Although this is a feature displayed by probably quite a number of excitable dynamical schemes when placed closer enough to oscillatory conditions, we have decided to investigate two very different examples of nonlinear models: the standard two variable FitzHugh-Nagumo (FHN) model and a single model incorporating a delay feedback. In both cases and by appropriately choosing the system parameters, we can separately control the time scales of both its excitatory and relaxational behaviors, favoring, in this way, the effect of fluctuations in tuning both coherence and anticoherence resonance.

Actually the use of feedback to either modify or control nonlinear dynamical responses has been proposed since a long time ago in different chemical [6] and biochemical [7] contexts. Much more recently, this question has been revisited after experiments have been conducted introducing global feedback techniques, either delayed [8] or nondelayed [9], into different scenarios of spatiotemporal pattern formation. Closer to our scenario, diverse situations of laser dynamics [10–12] have been explored, searching for the coherent role of the coupling of time delay and noise. From the theoretical point of view this question was already addressed in Refs. [13,14], and emphasizing features of resonant behavior in Refs. [15,16].

First and as a reference model, we consider the FHN model [2]

$$\varepsilon \dot{u} = u - \frac{u^3}{3} - v, \quad \dot{v} = u + h + \xi(t), \quad (2)$$

where  $u$  and  $v$  are the activator and inhibitor variables. For values of the parameter  $h$  slightly larger than one, the system is excitable but very close to the oscillatory regime.

The second model will exhibit more clearly the behavior we want to study. Here we will illustrate the subtle synergy

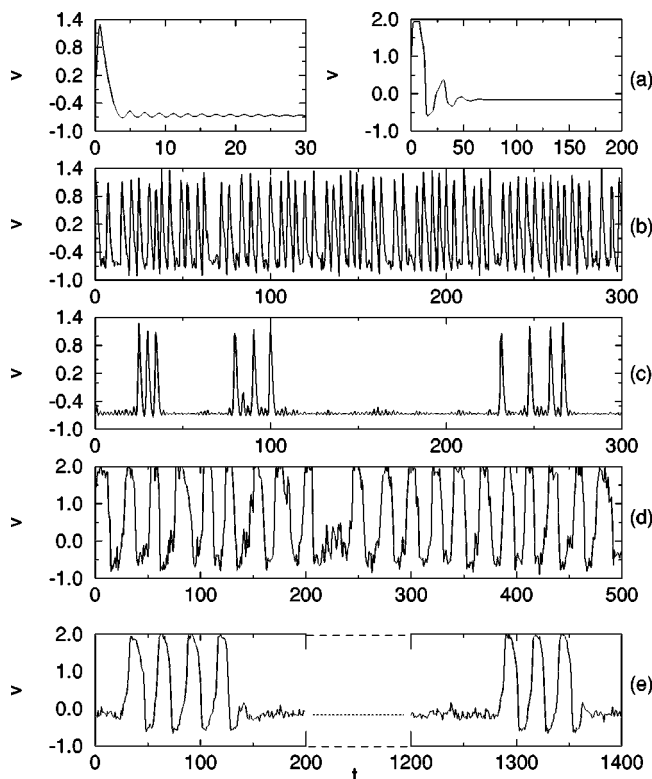


FIG. 1. A single pulse for the FHN model [(a), left] and for the feedback model [(a), right]. A train of pulses obtained from the FHN model for noise intensities  $D=0.02$  (b) and  $D=10^{-4}$  (c) and from the feedback model for noise intensities  $D=0.1$  (d) and  $D=7 \cdot 10^{-3}$  (e).

of noise and delay, that will lead to the previously anticipated anticonherence resonance phenomenon, by referring to a single variable bistable mechanism. As a prototype of a simple model with excitable characteristics, we present a one variable bistable system with a negative delay or feedback term. The explicit model equations read

$$\dot{v} = -v(v-a)(v-b) - c - \alpha v(t-T) + \xi(t), \quad (3)$$

where  $T$  is the time delay. The reaction term has three possible steady states and two of them are stable. The feedback term is linear and controls the stability of the steady states. This term acts as the inhibitor variable in standard excitable systems with a characteristic time  $T$ , acting here as a refractory time.

In both models the noise is prescribed to be Gaussian and white with an intensity  $D$ ,

$$\langle \xi(t)\xi(t') \rangle = 2D\delta(t-t'). \quad (4)$$

We have numerically integrated Eqs. (2) and (4) for parameters  $h=1.005$  and  $\varepsilon=0.1$ , and Eqs. (3) and (4) for parameters  $a=0.2$ ,  $b=2$ ,  $c=0.2$ ,  $\alpha=0.5$ , and  $T=8$ . These models have been defined to have dimensionless variables and parameters. In the absence of noise, these systems exhibit an excitable behavior in the form of pulses, with oscillating relaxations, as appear in Fig. 1(a) for both models. A pulse can be generated either by a special preparation of the

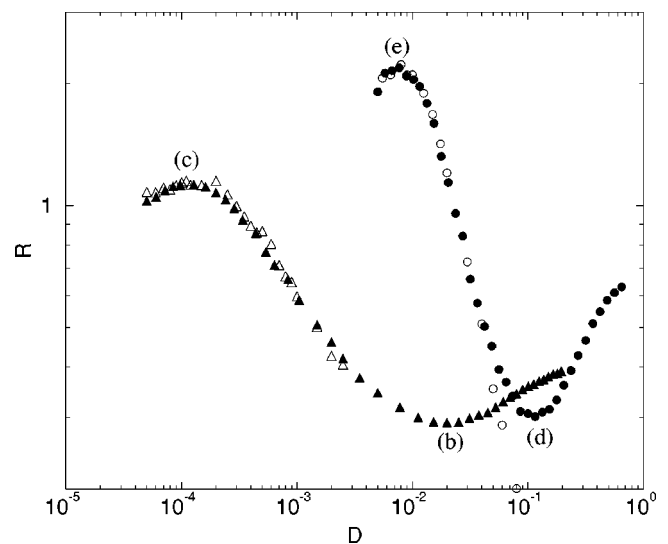


FIG. 2.  $R$  versus the noise intensity for the FHN model (filled triangles) and for the feedback model (filled circles). (b), (c), (d), and (e) correspond to those particular trajectories of Fig. 1. Open symbols correspond to results from our theoretical analysis (see text).

initial condition or by the presence of fluctuations. In the rest situation, the system is in a metastable state and even small fluctuations can allow the system to cross the potential barrier. Thus, in the presence of noise, a series of pulses, at more or less random intervals, can be observed. The statistics of these pulses are the subject of our study as a function of the noise intensity  $D$ . Let us analyze two runs of each model, with appropriately chosen and quite different noise intensities. We will start assuming that due to the oscillating decay of the pulse the next one to be generated will more likely take place at the time  $t_1$  (slightly larger than the intrinsic refractory time of the pulse) when this first maximum appears. Actually this is what one can see in Figs. 1(b) and 1(d). What is striking, however, is the completely different long time distribution of interpulse intervals in the extreme situations represented, respectively, by Figs. 1(c) and 1(e). In Figs. 1(b) and 1(d), we detect a clear signature of coherence resonance. Contrarily in Figs. 1(c) and 1(e), we observe again enchainned pulses grouping, this time, into small clusters separated by long and random time intervals.

This behavior is quantified in Fig. 2, where we show, for the two models, the numerical results of the coherence indicator  $R$  defined above, as a function of the noise intensity. First we observe a minimum of this quantity for some intermediate noise intensity. Such minima are indicative of the accepted signature of coherence resonance, that is, the presence of relatively coherent oscillations induced by a random perturbation of appropriate noise intensity [2]. Second and more importantly in Fig. 2, we observe the presence of a relative maximum, a feature which has not been reported in previous studies of coherence resonance. This maximum appears contrarily for very small values of the intensity of the noise, with the Poisson limit  $R=1$  reached from the above. Although the position of this maximum slightly depends on the particular model, its occurrence is a generic feature of the influence of the noise in both excitable systems, being more

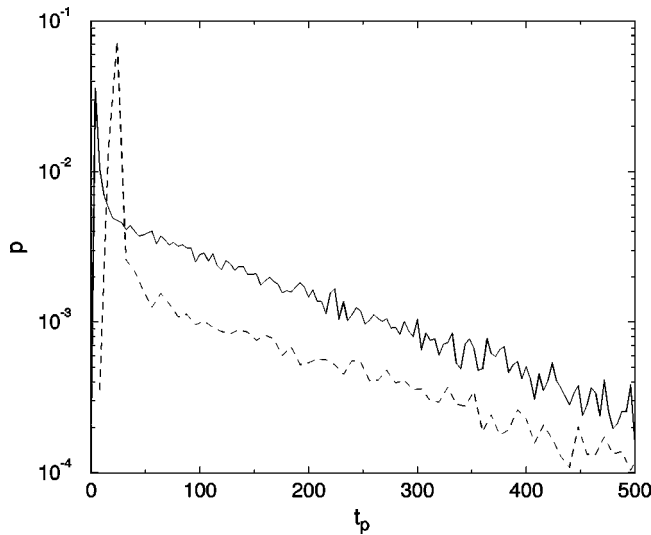


FIG. 3. Probability distributions of the interpulse time intervals for the FHN model (solid line) and for the feedback model (dashed line). They correspond to noise intensities  $D=10^{-4}$  and  $D=0.015$ , respectively.

pronounced in the model with delay.

Let us see how this maximum is possible from theoretical considerations based on the observation of the numerical data. According to the generic analysis of the simulation data we can conjecture the following functional behavior of the probability distribution of the time intervals between pulses: almost no pulses for times smaller than that corresponding to the first relaxational peak, a very pronounced maximum at this time and an exponential decaying distribution, characteristic of the Kramer's escape, for larger times.

We observe precisely this type of behavior (Fig. 3) from our numerical data accumulated in very large runs, and, accordingly, we propose the following probability distribution for the interpulse time intervals,

$$p(t_p) = A\delta(t_p - t_1) + B\theta(t_p - t_1)e^{-t_p/t_2}, \quad (5)$$

$t_1$  is associated with the characteristic time of the first maximum of the relaxational oscillations, and  $t_2$  is the characteristic time of the barrier crossing mechanism or Kramer's time.  $A$  and  $B$  are the relative weight parameters of the two functional components of  $p(t_p)$ . Due to the normalization of  $p(t_p)$  only one of them is independent and our choice here is to eliminate  $B$ . At this point our system has only three unknown parameters:  $t_1$ ,  $t_2$ , and  $A$ . An easy way to evaluate them from the simulation data is to look at the cumulative probability distribution,  $P(t_p) = \int_0^{t_p} p(t') dt'$ . This function is almost zero for small  $t_p$  with a high slope at  $t_1$ , and a relaxational approach to 1 for large  $t_p$ . Our numerical data follows quite well this functional dependence and moreover the value of  $t_1$  can be estimated easily at the point in which  $P(t_p)$  has a maximum slope. The values obtained are  $t_1 \sim 24$ , larger but proportional to  $T$  and independent of  $D$  for the feedback model and  $t_1 \sim 5$  for the FHN model. As  $P(t_p)$  has a simple exponential decay, we can also evaluate the other two parameters  $t_2(D)$  and  $A(D)$ .

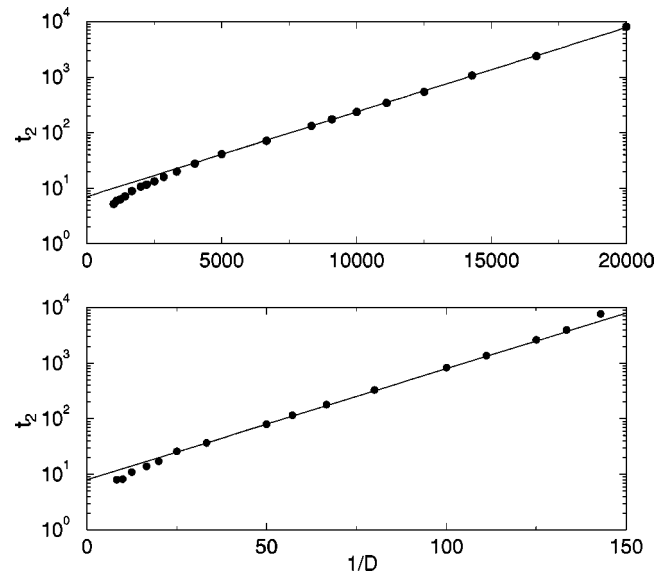


FIG. 4. Parameter  $t_2$  versus noise intensity for the FHN model (top) and the feedback model (bottom). Continuous lines are the fits given by Eq. (6), for the range of  $D$  values corresponding to  $R > 1$ .

The dependence of the parameters  $t_2$  on  $D$  for both models, is seen in Fig. 4. Assuming a Kramer's-like dependence of  $t_2$ ,

$$t_2 = t_0 e^{\Delta U/D}, \quad (6)$$

we obtain the parameter values  $t_0 \sim 8$  and  $\Delta U = 0.046$  for the feedback model and  $t_0 \sim 7$  and  $\Delta U = 0.00035$ , for the FHN model. The parameter  $A$  is similarly obtained although the explicit dependence on  $D$  is not simple but it can readily be tabulated.

Once the parameters of the system have been evaluated, we proceed with the calculation of the statistical moments. The first moment and the standard deviation of this distribution are

$$\begin{aligned} \langle t_p \rangle &= \int_0^\infty t_p p(t_p) dt_p = t_1 + (1-A)t_2, \\ \langle (\Delta t_p)^2 \rangle &= t_2^2 (1-A^2). \end{aligned} \quad (7)$$

From Eq. (7) the quantity  $R$ , defined in Eq. (1), can be written as

$$R = \frac{\sqrt{1-A^2}}{1+k-A}, \quad (8)$$

where  $k = t_1/t_2 < 1$ , compares these two time scales of the system.

In Fig. 2 we present a comparison between the numerical values of  $R$  versus  $D$  obtained from simulations and those obtained from our analysis using the parameters,  $t_1$ ,  $t_2(D)$  from Eq. (6), and  $A(D)$  from tabulated data. The agreement is remarkable in a wide range around the maximum in spite of the simplicity of the model.

According to our numerical estimates and for very small values of the noise intensity, we have that  $k \ll 1$  and consequently  $R > 1$ . Although the argument of Ref. [2], predicting  $R \leq 1$  when Kramer's barrier crossing mechanism dominates, is correct in most cases, this is not true if a new time scale is involved,  $t_1$ , which is precisely the situation near the oscillatory regime.

Our more physically oriented interpretation of the anticoherence phenomenon is thus the following. When adding additive noise of appropriately small intensity the system responds in what appears to be trains consisting of a few random number of enchain pulses with a deterministic time scale separation  $t_1$ . These clustering episodes appear, however, in a considerably unpredictable way separated by long time intervals  $t_2$ , when recording the whole signal over long times. This is the signature of maximal anticoherence: the mixture of two very different time scales. In this regime the variance  $\langle (\Delta t_p)^2 \rangle$  decreases with  $D$  but  $\langle t_p \rangle$  decreases much more rapidly, producing the relative enhancement of  $R$ .

Two very different dynamical models show the same trends and one can thus conclude that this as a universal and robust phenomena induced by noise and characteristic of excitable systems near its oscillatory regime, where the relaxation to the steady state is not monotonous but oscillatory

leading to a dynamical excitability threshold. The anticoherence phenomenon for very small values of the intensity of the noise is also implicit in the figures of Refs. [11,12].

What is also remarkable, although this is indeed not totally unexpected, is that features of coherence resonance are also captured at larger noise intensities. In summary, we observe a crossover from maximal anticoherence to maximal coherence as noise intensity increases. In the context of stochastic resonance, a similar crossover has been found in Ref. [17]. The mechanism opposite to that of standard stochastic resonance is termed resonant trapping.

Finally let us emphasize that to convince ourselves that such combined effects of coherent and anticoherent resonance are not at all spurious effects of using uncorrelated fluctuations (white noise), we have checked that the behavior of  $R$  shown in Fig. 2 also appears for coloured noises (results not shown here).

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- [1] L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchesoni, *Rev. Mod. Phys.* **70**, 223 (1998).
- [2] A.S. Pikovsky and J. Kurths, *Phys. Rev. Lett.* **78**, 775 (1997).
- [3] D.E. Postnov *et al.*, *Phys. Rev. E* **59**, R3791 (1999); A. Neiman *et al.*, *Phys. Rev. Lett.* **83**, 4896 (1999); B. Hu and C. Zhou, *Phys. Rev. E* **61**, R1001 (2000); C. Zhou, J. Kurths, and B. Hu, *Phys. Rev. Lett.* **87**, 098101 (2001).
- [4] W.J. Wilbur and J. Rinzel, *J. Theor. Biol.* **105**, 345 (1983).
- [5] C. Koch, *Biophysics of Computation* (Oxford University Press, New York, 1999).
- [6] I. Epstein, *J. Chem. Phys.* **92**, 1702 (1990).
- [7] P. Rapp, *Faraday Symp. Chem. Soc.* **9**, 215 (1974).
- [8] M. Kim, M. Bertram, M. Pollmann, A. von Oertzen, A.S. Mikhailov, H.H. Rotermund, and G. Ertl, *Science* **292**, 1357 (2001).
- [9] V.K. Vanag, L. Yang, M. Dolnik, A.M. Zhabotinsky, and I.R. Epstein, *Nature (London)* **406**, 389 (2000).
- [10] G. Giacomelli, M. Giudici, S. Balle, and J.R. Tredicce, *Phys. Rev. Lett.* **84**, 3298 (2000).
- [11] J.M. Buldú, J. García-Ojalvo, C.R. Mirasso, M.C. Torrent, and J.M. Sancho, *Phys. Rev. E* **64**, 051109 (2001).
- [12] C. Masoller, *Phys. Rev. Lett.* **88**, 034102 (2002).
- [13] A. Longtin, J.G. Milton, J.E. Bos, and M.C. Mackey, *Phys. Rev. A* **41**, 6992 (1990).
- [14] S. Guillouzie, I. L'Heureux, and A. Longtin, *Phys. Rev. E* **59**, 3970 (1999); **61**, 4906 (2000).
- [15] T. Ohira and Y. Sato, *Phys. Rev. Lett.* **82**, 2811 (1999).
- [16] L.S. Tsimring and A. Pikovsky, *Phys. Rev. Lett.* **87**, 250602 (2001).
- [17] F. Apostolico, L. Gammaitoni, F. Marchesoni, and S. Santucci, *Phys. Rev. E* **55**, 36 (1997).